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Asymptotic analysis for the Cauchy problem for a functional PDE

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1. Introduction

We consider the asymptotic bahavior of solutions of the Cauchy problem for the functional partial differential equation

$$
(\mathrm{CP})_\varepsilon \qquad \begin{cases} (\mathrm{E})_\varepsilon \quad u_t^\varepsilon(x,t,\xi) = \dfrac{1}{\varepsilon} H(Du^\varepsilon(x,t,\xi),\xi) \\ \qquad\qquad + \dfrac{1}{\varepsilon^2}\displaystyle\int_I k(\xi,\eta)\,[u^\varepsilon(x,t,\eta)-u^\varepsilon(x,t,\xi)]\,d\eta \\ \qquad\qquad \text{for } (x,t,\xi)\in\mathbf{R}^n\times(0,\infty)\times I, \\ u^\varepsilon(x,0,\xi)=g(x,\xi) \quad \text{for } (x,\xi)\in\mathbf{R}^n\times I, \end{cases}
$$

where ε is a positive parameter, I is a given finite interval of the real line, H is a Borel function on $\mathbf{R}^n \times I$ such that for each $\xi \in I$ the function $H(\cdot,\xi)$ is continuous on $\mathbf{R}^n$, and k is a bounded, positive, Borel measurable function on $I \times I$.

The functional partial differential equation $(\mathrm{E})_\varepsilon$ may be regarded as an infinite system of first order partial differential equations. Indeed, one of our motivations to study $(\mathrm{CP})_\varepsilon$ is to extend an asymptotic result obtained in Evans [3] for a finite system of partial differential equations to that for $(\mathrm{CP})_\varepsilon$. Prior to [3] there are many contributions to the asymptotic behavior of solutions of systems of differential equations related to the problems treated in [3] and we refer for these to [3], [6], [7] and the references therein.

The functional partial differential equation $(\mathrm{E})_\varepsilon$ arises as a fundamental equation for the optimal control of the system whose states are described by ordinary differential equations, subject to random changes of states in I and to control which induce the integral term in $(\mathrm{E})_\varepsilon$ and the nonlinearity of H, respectively.

Other than the extension to infinite systems, new features in this paper beyond [3] are: (i) the treatment of the initial layer, i.e., the case when the initial data $g(x,\xi)$ depends on ξ and (ii) the nonlinearity of the term H.

In our asymptotic analysis of $(\mathrm{CP})_\varepsilon$, we use the perturbed test function method developed in [3], which is based on the notion of viscosity solution and the stability properties of viscosity solutions. The extension from finite systems to infinite systems was not trivial and, as we will see in section 3, we need to take into account of terms up to order ε^2 when we build the perturbed test function.

The problem of the initial layer in our analysis is resolved by constructing appropriate barrier functions, a result of which is stated in Lemma 3.4 below. On the other hand, the extension to the nonlinear term H is rather straightforward.

2. Preliminaries

We use the following notation: $Q_T = \mathbf{R}^n \times (0, T)$, $R_T = \mathbf{R}^n \times [0, T)$ for $0 < T \leq \infty$, and for function $f : S \to \mathbf{R}^m$ we write $\|f\|_\infty = \sup_S |f|$. I denotes a fixed finite interval, with length $|I| > 0$, and also the identity operator on a given space.

For any $k \in \mathbf{Z}_+ := \mathbf{N} \cup \{0\}$ and $\Omega \subset \mathbf{R}^m$, $C^k(\Omega) \otimes \mathcal{B}(I)$ denotes the set of functions f on $\Omega \times I$ such that for each $x \in \Omega$ the function $f(x, \cdot)$ is Borel measurable in I and for each $\xi \in I$ the function $f(\cdot, \xi)$ is k times continuously differentiable on Ω. We write also $C(\Omega) \otimes \mathcal{B}(I)$ for $C^0(\Omega) \otimes \mathcal{B}(I)$. For any Borel subset $\Omega \subset \mathbf{R}^m$, $\mathcal{B}(\Omega)$ denotes the space of all Borel functions on Ω, and $\mathcal{B}^\infty(\Omega)$ denotes the Banach space of bounded Borel functions f on Ω with norm $\|f\|_\infty$.

Throughout this paper we fix positive numbers κ_0, κ_1, with $\kappa_0 < \kappa_1$, and consider the class $\mathcal{D}_0$ of Borel functions k on $I \times I$ such that $\kappa_0 \leq k(\xi, \eta) \leq \kappa_1$ for all $\xi, \eta \in I$.

We call a continuous function $\omega : [0, \infty) \to [0, \infty)$ a modulus if ω is non-decreasing in $[0, \infty)$ and $\omega(0) = 0$.

Let G_1 and G_2 denote the sets, respectively, of all pairs (ω, L) of a modulus ω and a positive constant L and of all pairs of a collection $\{\omega_R\}_{R>0}$ of moduli and a collection $\{L_R\}_{R>0}$ of positive constants. We write $G = G_1 \times G_2$.

For $\gamma_1 \equiv (\omega, L) \in G_1$ let $\mathcal{D}_1(\gamma_1)$ denote the set of all functions $g \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$\text{(D1)} \qquad |g(x,\xi) - g(y,\xi)| \leq \omega(|x-y|), \quad |g(x,\xi)| \leq L \quad \text{for all } x, y \in \mathbf{R}^n,\ \xi \in I.$$

For $\gamma_2 \equiv (\{\omega_R\}_{R>0}, \{L_R\}_{R>0}) \in G_2$ let $\mathcal{D}_2(\gamma_2)$ denote the set of all functions $H \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$\text{(D2)} \qquad \begin{aligned} &|H(p,\xi) - H(q,\xi)| \leq \omega_R(|p-q|), \quad |H(p,\xi)| \leq L_R \\ &\qquad\qquad \text{for all } p, q \in B(0,R),\ \xi \in I,\ R > 0, \end{aligned}$$

where $B(0, R)$ denotes the closed ball with radius R centered at the origin. For $\gamma \equiv (\gamma_1, \gamma_2) \in G$ we write

$$\mathcal{D}(\gamma) = \mathcal{D}_0 \times \mathcal{D}_1(\gamma_1) \times \mathcal{D}_2(\gamma_2),$$

and set

$$\mathcal{D}_i = \bigcup \{\mathcal{D}_i(\gamma) \mid \gamma \in G_i\} \quad \text{for } i = 1, 2 \quad \text{and} \quad \mathcal{D} = \bigcup \{\mathcal{D}(\gamma) \mid \gamma \in G\}.$$

We often consider the subclass of functions $k \in \mathcal{D}_0$ for which

$$\text{(K1)} \qquad \int_I k(\xi, \eta)\, d\eta = 1 \quad \text{for all } \xi \in I.$$

For such a function k, we define the continuous linear operator $K : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ by

$$\text{(2.1)} \qquad Kf(\xi) = \int_I k(\xi, \eta) f(\eta)\, d\eta \quad \text{for } \xi \in I.$$

Note that this formula extends the domain of definition of K to the space of (Lebesgue) measurable functions $f : I \to \mathbf{R}$ which are integrable. Associated with this operator, we define the compact linear operator $\bar{K} : L^2(I) \to L^2(I)$ by

$$\bar{K}f(\xi) = \int_I k(\xi,\eta)f(\eta)d\eta \quad \text{for } f \in L^2(I). \tag{2.2}$$

As usual and in the above formula, we often identify elements of $L^2(I)$ with measurable functions on I, the square of which are integrable. The precise meaning of (2.2) is the following: for function $f : I \to \mathbf{R}$ which is measurable and such that $|f|^2$ is integrable, let

$$[f] := \{g : I \to \mathbf{R} \mid g \text{ measurable, } g(\xi) = f(\xi) \text{ a.e. } \xi \in I\}.$$

With this notation, $\bar{K}$ is defined by

$$\bar{K}[f] = [Kf].$$

By hypothesis (K1), the operator $\bar{K}$ has unity as its eigenvalue and the function $\mathbf{1} \in L^2(I)$ defined by $\mathbf{1}(\xi) \equiv 1$ as a corresponding eigenfunction. By the Perron-Frobenius theory, we see that the kernel $\mathrm{Ker}\,(I - \bar{K})$ is one-dimensional, i.e.,

$$\mathrm{Ker}\,(I - \bar{K}) = \mathrm{span}\,\{\mathbf{1}\}.$$

(See the proof of Lemma 3.5 in section 3.)

By the Fredholm-Riesz-Schauder theory (see, e.g., [8]), the kernel $\mathrm{Ker}\,(I - \bar{K}^*)$, where $\bar{K}^*$ denotes the adjoint operator of $\bar{K}$, is a one–dimensional subspace of $L^2(I)$. Hence, there exists a unique vector $r \in L^2(I)$ such that

$$\int_I r(\xi)k(\xi,\eta)d\xi = r(\eta) \quad \text{a.e. } \eta \in I,$$
$$\int_I r(\xi)d\xi = 1.$$

When we regard the vector r as a function, we may assume by replacing r if necessary that $r \in \mathcal{B}^\infty(I)$ and that

$$\int_I r(\xi)k(\xi,\eta)d\xi = r(\eta) \quad \text{for all } \eta \in I. \tag{2.3}$$

Moreover, by the Perron-Frobenius theory, we see that $r(\xi) > 0$ for all $\xi \in I$. Then form (2.3) we get

$$\kappa_0 \le r(\xi) \le \kappa_1 \quad \text{for } \xi \in I. \tag{2.4}$$

By the Fredholm-Riesz-Schauder theory, there is a bounded linear operator $\bar{S} : \{r\}^\perp \to \{\mathbf{1}\}^\perp$, where $B^\perp$ denotes the orthogonal complement of B in $L^2(I)$, such that

$$\bar{S}f - \bar{K}\bar{S}f = f \quad \text{for } f \in \{r\}^\perp. \tag{2.5}$$

For any integrable function $h : I \to \mathbf{R}$, we define

$$\{h\}^{\perp,\infty} = \{f \in \mathcal{B}^\infty(I) \mid \int_I h(\xi)f(\xi)d\xi = 0\}.$$

Associated with $\bar{S}$, we define a continuous linear operator $S : \{r\}^{\perp,\infty} \to \{\mathbf{1}\}^{\perp,\infty}$ by

$$Sf = f + Kg, \qquad \text{with } g \in \bar{S}[f].$$

Here note that Kg does not depend on the choice of $g \in \bar{S}[f]$ and that for $f \in \{r\}^{\perp,\infty}$ and $g \in \bar{S}[f]$,

$$\begin{aligned} |Kg(\xi)| &\le \int_I k(\xi,\eta)|g(\eta)|d\eta \le \left(\int_I |k(\xi,\eta)|^2 d\eta\right)^{1/2} \|\bar{S}[f]\|_2 \\ &\le \kappa_1 |I|^{1/2} \|\bar{S}\| \|f\|_2 \le \kappa_1 |I| \|\bar{S}\| \|f\|_\infty, \end{aligned}$$

where $\|f\|_2 = (\int_I |f(\xi)|^2 d\xi)^{1/2}$.

Now, (2.5) reads

$$(I - K)Sf = f \quad \text{for } f \in \{r\}^{\perp,\infty}. \tag{2.6}$$

Let $H \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ satisfy (D2) for some $(\{\omega_R\}, \{L_R\}) \in G_2$ and

$$\int_I H(p,\xi) r(\xi) d\xi = 0 \quad \text{for } p \in \mathbf{R}^n. \tag{H1}$$

We define $a \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ by

$$a(p,\cdot) = SH(p,\cdot). \tag{2.7}$$

Observe that if, in addition, we assume that $H \in C^m(\mathbf{R}^n) \otimes \mathcal{B}(I)$ for some $m \in \mathbf{N}$ and that for each $R > 0$ there are a constant $C_R > 0$ and a modulus ω_R such that for any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}^n_+$ with $\alpha_1 + \cdots \alpha_n \le m$,

(2.8)

$$\begin{aligned} &|D^\alpha_p H(p,\xi)| \le C_R \quad \text{for } (p,\xi) \in B(0,R) \times I,\ \xi \in I, \\ &|D^\alpha_p H(p,\xi) - D^\alpha_p H(q,\xi)| \le \omega_R(|p-q|) \quad \text{for } p,q \in B(0,R),\ \xi \in I,\ R > 0, \end{aligned}$$

and if we set $f(p,\xi) = SH(p,\cdot)(\xi)$ for $(p,\xi) \in \mathbf{R}^n \times I$, then $f \in C^m(\mathbf{R}^n) \otimes \mathcal{B}(I)$ and furthermore for each $R > 0$ there exist a constant $M_R > 0$ and a modulus μ_R such that for any multi-index $\alpha \in \mathbf{Z}^n_+$, with $\alpha_1 + \cdots + \alpha_n \le m$,

$$\begin{aligned} &|D^\alpha_p f(p,\xi)| \le M_R \quad \text{for } (p,\xi) \in B(0,R) \times I,\ R > 0, \\ &|D^\alpha_p f(p,\xi) - D^\alpha_p f(q,\xi)| \le \mu_R(|p-q|) \quad \text{for } p,q \in B(0,R),\ \xi \in I,\ R > 0. \end{aligned} \tag{2.9}$$

In addition to (D2) and (H1), we assume that $H \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$ and that H satisfies (2.8) with $m = 1$. We define $A : \mathbf{R}^n \times I \to \mathcal{S}^n$ and $\bar{A} : \mathbf{R}^n \to \mathcal{S}^n$ by

$$\begin{aligned} A(p,\xi) &= \frac{1}{2}\left(D_p H(p,\xi) \otimes D_p a(p,\xi) + D_p a(p,\xi) \otimes D_p H(p,\xi)\right), \\ \bar{A}(p) &= \int_I r(\xi) A(p,\xi) d\xi. \end{aligned} \tag{2.10}$$

The components of the matrix-valued function A belong to $C(\mathbf{R}^n)\otimes\mathcal{B}(I)$. Also, in view of (2.9), we see that $\bar{A}$ is continuous on $\mathbf{R}^n$. We claim that $\bar{A}(p)$ is a non-negative definite matrix for any $p\in\mathbf{R}^n$. To see this, we first observe that

$$D_pH(p,\xi)=D_pa(p,\xi)-\int_I k(\xi,\eta)D_pa(p,\eta)d\eta \quad \text{for all } (p,\xi)\in\mathbf{R}^n\times I.$$

Let $y\in\mathbf{R}^n$ and compute that for $p\in\mathbf{R}^n$,

$$\begin{aligned}
&\langle\bar{A}(p)y,y\rangle\\
&=\int_I r(\xi)\langle D_pH(p,\xi),y\rangle\langle D_pa(p,\xi),y\rangle d\xi\\
&=\int_I r(\xi)\langle D_pa(p,\xi),y\rangle^2 d\xi-\iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\eta),y\rangle\langle D_pa(p,\xi),y\rangle d\xi d\eta\\
&\geq\int_I r(\xi)\langle D_pa(p,\xi),y\rangle^2 d\xi\\
&-\Big(\iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\eta),y\rangle^2 d\xi d\eta\Big)^{1/2}\Big(\iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\xi),y\rangle^2 d\xi d\eta\Big)^{1/2}\\
&=0,
\end{aligned}$$

which was to be proven. Here and henceforth we write $\langle p,q\rangle$ for the Euclidean inner product of $p,q\in\mathbf{R}^n$.

Let $\Omega\subset R_\infty$ and $(\nu,M)\in G_1$. We denote by $\mathcal{U}(\nu,M)\equiv\mathcal{U}(\Omega\times I;\nu,M)$ the set of functions $u\in C(\Omega)\otimes\mathcal{B}(I)$ such that

$$\begin{aligned}
&|u(x,t,\xi)-u(y,s,\xi)|\leq\nu(|x-y|+|t-s|)\\
&|u(x,0,\xi)|\leq M
\end{aligned}$$

for all $(x,t)\in\Omega$ and $\xi\in I$. We denote

$$\mathcal{U}\equiv\mathcal{U}(\Omega\times I)=\bigcup\{\mathcal{U}(\lambda)\mid\lambda\in G_1\}.$$

We write

$$\mathcal{U}_c(\Omega\times I;\lambda)=\mathcal{U}(\Omega\times I;\lambda)\cap C(\Omega\times I),\quad \mathcal{U}_c(\Omega\times I)=\mathcal{U}(\Omega\times I)\cap C(\Omega\times I).$$

We denote by $\mathcal{U}^+(\Omega\times I)$ the set of those functions u on $\Omega\times I$ such that for each $(x,t)\in\Omega$ the function $u(x,t,\cdot)$ is Borel measurable and integrable in I and for each $\xi\in I$ the function $u(\cdot,\xi)$ is upper semicontinuous in Ω. We set $\mathcal{U}^-(\Omega\times I)=-\mathcal{U}^+(\Omega\times I)$.

Next, we give the definition of viscosity solutions of

$$\text{(E)}\qquad \begin{aligned} u_t(x,t,\xi)\ &=\ H(Du(x,t,\xi),\xi)+\int_I k(\xi,\eta)\,[u(x,t,\eta)-u(x,t,\xi)]\,d\eta\\ &\qquad\text{for } (x,t,\xi)\in\mathbf{R}^n\times(0,\infty)\times I,\end{aligned}$$

Definition. *Let $\Omega\subset Q_\infty$ be an open subset and $(k,H)\in\mathcal{D}_0\times\mathcal{D}_2$.* (i) *We call $u\in\mathcal{U}^+(\Omega\times I)$ a viscosity subsolution of* (E) *in $\Omega\times I$ if whenever $\varphi\in C^1(\Omega)$, $\xi\in I$, and $u(\cdot,\xi)-\varphi$ attains its local maximum at $(\hat{x},\hat{t})$, then*

$$\varphi_t(\hat{x},\hat{t})\leq H(D\varphi(\hat{x},\hat{t}),\xi)+\int_I k(\xi,\eta)[u(\hat{x},\hat{t},\eta)-u(\hat{x},\hat{t},\xi)]d\eta.$$

(ii) *Similarly we call* $u \in \mathcal{U}^-(\Omega \times I)$ *a viscosity supersolution of* (E) *in* $\Omega \times I$ *if whenever* $\varphi \in C^1(\Omega)$, $\xi \in I$, *and* $u(\cdot, \xi) - \varphi$ *attains its local minimum at* $(\hat{x}, \hat{t})$, *then*

$$\varphi_t(\hat{x}, \hat{t}) \geq H(D\varphi(\hat{x}, \hat{t}), \xi) + \int_I k(\xi, \eta)[u(\hat{x}, \hat{t}, \eta) - u(\hat{x}, \hat{t}, \xi)]d\eta.$$

(iii) *Finally, we call* $u \in C(\Omega) \otimes \mathcal{B}(I)$ *a viscosity solution of* (E) *in* $\Omega \times I$ *if it is both a viscosity sub- and supersolution of* (E) *in* $\Omega \times I$.

For the definition of viscosity solutions of $(\mathrm{E})_0$, we use the standard definition, for which we refer to [1].

3. Main results

Theorem 3.1. *Let* $(k, g, H) \in \mathcal{D}$. *Then there is a unique viscosity solution* $u \in \mathcal{U}(R_\infty \times I)$ *of* $(\mathrm{CP})_\varepsilon$.

If k is continuous on $I \times I$, then the proof of Theorem 3.1 is standard. (See [4].) In case that k is Borel measurable on $I \times I$, we use an argument based on monotone classes of functions. (See [5].)

Of course, $u \in \mathcal{U}(R_\infty \times I)$ is defined to be a viscosity solution of $(\mathrm{CP})_\varepsilon$ if it is a viscosity solution of $(\mathrm{E})_\varepsilon$ in $Q_\infty \times I$ and it satisfies the initial condition: $u^\varepsilon(x, 0, \xi) = g(x, \xi)$ for all $(x, \xi) \in \mathbf{R}^n \times I$.

Theorem 3.2. *Let* $k \in \mathcal{D}_0$, $g \in \mathrm{BUC}(\mathbf{R}^n)$, *and* $H \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$. *Assume that* (K1) *and* (H1) *hold and that* (2.8), *with* $m = 1$, *hold for some* $(\{\omega_R\}, \{C_R\}) \in G_2$. *Then there is a unique viscosity solution* $u \in \mathrm{BUC}(R_\infty)$ *of*

$$(\mathrm{CP})_0 \qquad \begin{cases} (\mathrm{E})_0 \quad u_t(x,t) = \operatorname{tr}[\bar{A}(Du(x,t))D^2u(x,t)] \\ \qquad\qquad\qquad\qquad \text{for } (x,t) \in \mathbf{R}^n \times (0, \infty), \\ \qquad\quad u(x, 0) = g(x) \quad \text{for } x \in \mathbf{R}^n. \end{cases}$$

See [1] for the proof of Theorem.

The assumptions on k and H in the above theorem are made just to make sure that the function $\bar{A}$ is continuous on $\mathbf{R}^n$.

Theorem 3.3. *Let* $(k, g, H) \in \mathcal{D}$. *Assume that* (K1) *and* (H1) *hold and that* H *satisfies* (2.8), *with* $m = 1$, *for some* $(\{\omega_R\}, \{C_R\}) \in G_2$. *Set*

$$\bar{g}(x) = \int_I r(\xi) g(x, \xi) d\xi \quad \text{for } x \in \mathbf{R}^n.$$

Let $u^\varepsilon \in \mathcal{U}(R_\infty \times I)$ *be the viscosity solution of* $(\mathrm{CP})_\varepsilon$. *Let* $u \in \mathrm{BUC}(R_\infty)$ *be the viscosity solution of* $(\mathrm{CP})_0$ *with* $\bar{g}$ *in place of* g. *Then, for each* $\delta \in (0, 1)$,

$$\lim_{\varepsilon \searrow 0} \sup\{|u^\varepsilon(x, t, \xi) - u(x, t)| \mid (x, t, \xi) \in \mathbf{R}^n \times [\delta, \delta^{-1}] \times I\} = 0.$$

In addition, if $g(x,\xi)$ is independent of ξ, then for each $T>0$

$$\lim_{\varepsilon\searrow 0}\sup\{|u^\varepsilon(x,t,\xi)-u(x,t)| \mid (x,t,\xi)\in \mathbf{R}^n\times[0,T]\times I\}=0.$$

We introduce only the proof of Theorem 3.3.

Let $(k,g,H)\in\mathcal{D}$, $\bar g$, $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$, and u be as in Theorem 3.3.

Note that, by (K1), $(\mathrm{E})_\varepsilon$ reads

$$u_t^\varepsilon(x,t,\xi)=\frac{1}{\varepsilon}H(Du^\varepsilon(x,t,\xi),\xi)+\frac{1}{\varepsilon^2}\Big(\int_I k(\xi,\eta)u^\varepsilon(x,t,\eta)d\eta-u^\varepsilon(x,t,\xi)\Big)$$
$$\text{for } (x,t,\xi)\in\mathbf{R}^n\times(0,\infty)\times I.$$

We set $h(x,\xi)=g(x,\xi)-\bar g(x)$ for $(x,\xi)\in\mathbf{R}^n\times I$, and note that

$$\int_I r(\xi)h(x,\xi)d\xi=0\qquad\text{for all } x\in\mathbf{R}^n.$$

To prove Theorem 3.3, we use the so-called relaxed limits. We define

$$u^+(x,t)=\lim\nolimits_{r\searrow 0}\sup\{u^\varepsilon(y,s,\eta)\mid(y,s,\eta)\in R_\infty\times I,|y-x|+|s-t|<r\}$$
$$u^-(x,t)=\lim\nolimits_{r\searrow 0}\inf\{u^\varepsilon(y,s,\eta)\mid(y,s,\eta)\in R_\infty\times I,|y-x|+|s-t|<r\}$$

for $(x,t)\in R_\infty\times I$.

Lemma 3.4. *There is a modulus μ such that*

$$\bar g(x)-\mu(t)\le u^-(x,t)\le u^+(x,t)\le\bar g(x)+\mu(t)\qquad\textit{for } (x,t)\in Q_\infty.$$

In addition, if $h=0$, then the above inequalities hold for all $(x,t)\in R_\infty$.

Lemma 3.5. *There are constants $\delta>0$ and $C_0>0$ such that for any $h\in\{r\}^{\perp,\infty}$,*

$$\|e^{t(K-I)}h\|_\infty\le C_0e^{-\delta t}\|h\|_\infty\qquad\textit{for all } t\ge 0.$$

Proof of Lemma 3.5. In this proof we regard $L^2(I)$, $\mathcal{B}^\infty(I)$, etc. as the vector spaces with complex scalar field.

We first prove that

(3.1) if $\mu\in\mathbf{C}$ is an eigenvalue of $\bar K$ and $|\mu|\ge 1$, then $\mu=1$.

To show this, we fix $\mu\in\mathbf{C}$ and $\phi\in L^2(I)$ so that $|\mu|\ge 1$, $\phi\ne 0$, and $\bar K\phi=\mu\phi$.

Identifying ϕ with the function h defined by

$$h(\xi)=\mu^{-1}\int_I k(\xi,\eta)g(\eta)d\eta,$$

where g is a function in the equivalence class ϕ, we may regard ϕ as a function in $\mathcal{B}^\infty(I)$ and assume that

$$\mu\phi(\xi)=K\phi(\xi)\qquad\text{for all } \xi\in I.$$

Set $M = \sup_I |\phi|$. We claim that $|\phi(\xi)| = M$ a.e. $\xi \in I$. In order to check this, we fix $\varepsilon > 0$ and $\gamma > \varepsilon$, and choose $\xi \in I$ so that $|\phi(\xi)| > M - \varepsilon$. Observing that $|\phi(\xi)| \le \bar{K}|\phi|(\xi)$ and setting $B_\gamma = \{\xi \in I \mid |\phi(\xi)| \le M - \gamma\}$, we calculate that

$$\begin{aligned} 0 &< \int_I k(\xi,\eta)(|\phi(\eta)| - M + \varepsilon)d\eta \\ &\le \int_{B_\gamma} k(\xi,\eta)(\varepsilon - \gamma)d\eta + \int_I k(\xi,\eta)\varepsilon d\eta \le -(\gamma - \varepsilon)\kappa_0|B_\gamma| + \varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0$, we see that $|B_\gamma| = 0$ for all $\gamma > 0$, which shows that $|\phi(\xi)| = M$ a.e. $\xi \in I$.

By multiplying ϕ by M^{-1} if necessary, we may assume that $M = 1$. We fix $\hat{\xi} \in I$ so that $|\phi(\hat{\xi})| = 1$. We may assume by multiplying ϕ by $\overline{\phi(\hat{\xi})}$, the complex conjugate of $\phi(\hat{\xi})$, that $\phi(\hat{\xi}) = 1$. Define $a \in \mathcal{B}(I)$ by $a(\xi) = \operatorname{Re}\phi(\xi)$. It follows that $a(\hat{\xi}) = 1$ and $|a(\xi)| \le 1$ for all $\xi \in I$. Setting $B_\varepsilon = \{\xi \in I \mid a(\xi) \le 1 - \varepsilon\}$ for $\varepsilon > 0$, we argue as before, to get

$$0 \le -\varepsilon \int_{B_\varepsilon} k(\hat{\xi},\eta)d\eta \le -\varepsilon\kappa_0|B_\varepsilon|,$$

which guarantees that $\phi(\xi) = 1$ a.e. $\xi \in I$. Thus we have

$$\mu\phi(\xi) = K\phi(\xi) = 1 \quad \text{for } \xi \in I,$$

and conclude that $\mu = 1$ and $\phi(\xi) = 1$ for all $\xi \in I$.

Next, we observe that for $\phi \in \{r\}^\perp$,

$$\int_I \bar{K}\phi(\xi)r(\xi)d\xi = \int_I \phi(\xi)\bar{K}^* r(\xi)d\xi = \int_I \phi(\xi)r(\xi)d\xi = 0. \tag{3.2}$$

This allows us to define the continuous linear operator $\bar{L} : \{r\}^\perp \to \{r\}^\perp$ by $\bar{L}\phi = \bar{K}\phi$.

Since $\bar{K}$ is a compact operator on $L^2(I)$, we see that $\bar{L}$ is a compact operator on $\{r\}^\perp$. By the Fredholm-Riesz-Schauder theory, we know that for each $\varepsilon > 0$, $\sigma(\bar{L}) \cap \{z \in \mathbf{C} \mid |z| > \varepsilon\}$ is a finite set and consists of eigenvalues of $\bar{L}$. Here and henceforth, for any operator L, $\sigma(L)$ denotes the spectrum of L. Since $\mathbf{1} \notin \{r\}^\perp$, we see from (3.1) that $\sigma(\bar{L}) \subset \{z \in \mathbf{C} \mid |z| < 1\}$. Since $\sigma(\bar{L})$ is a closed subset of $\mathbf{C}$, we find a constant $\theta \in (0,1)$ such that

$$\sigma(\bar{L}) \subset \{z \in \mathbf{C} \mid |z| \le \theta\}. \tag{3.3}$$

In view of (3.2), we may define the continuous operator $L : \{r\}^{\perp,\infty} \to \{r\}^{\perp,\infty}$ by $L\phi = K\phi$. We claim that

$$\sigma(L) \subset \{z \in \mathbf{C} \mid |z| \le \theta\}. \tag{3.4}$$

To show this, fix $\mu \in \{z \in \mathbf{C} \mid |z| > \theta\}$. For $\phi \in \{r\}^{\perp,\infty}$ choose any

$$\psi \in (\mu I - \bar{L})^{-1}[\phi],$$

and set

$$f(\xi) = \mu^{-1}(K\psi(\xi) - \phi(\xi)) \quad \text{for } \xi \in I.$$

It is easily seen that $f \in \{r\}^{\perp,\infty}$ and that

$$\mu f(\xi) - Lf(\xi) = \phi(\xi) \quad \text{for all } \xi \in I.$$

Hence, $\mu I - L$ is surjective. Next we fix $\phi \in \{r\}^{\perp,\infty}$. Let $f, g \in \{r\}^{\perp,\infty}$ satisfy

$$(\mu I - L)f(\xi) = \phi(\xi) \quad \text{and} \quad (\mu I - L)g(\xi) = \phi(\xi) \quad \text{for } \xi \in I.$$

Then we see that $[f - g] \in \operatorname{Ker}(\mu I - \bar{L})$, which yields in view of (3.3) that $f(\xi) = g(\xi)$ a.e. $\xi \in I$. Accordingly we have

$$\mu(f - g)(\xi) = L(f - g)(\xi) = 0 \quad \text{for } \xi \in I.$$

Thus $\mu I - L$ is injective. Invoking the open mapping theorem, we conclude that μ is in the resolvent set of L, proving (3.4).

Recall the definition of the spectral radius ρ of the operator L, i.e.,

$$\rho = \lim_{k\to\infty} \|L^k\|^{1/k}.$$

(See [8].) We know that $\rho \leq \theta$. Fix any $\lambda \in (\theta, 1)$. Then there is a constant $C \geq 1$ such that

$$\|L^k\| \leq C\lambda^k \quad \text{for all } k \in \mathbf{N}.$$

This yields that for $t \geq 0$,

$$\|e^{tL}\| \leq \sum_{k\in\mathbf{Z}_+} \frac{t^k\|L^k\|}{k!} \leq Ce^{\lambda t}.$$

Thus, for $h \in \{r\}^{\perp,\infty}$ and $t \geq 0$ we have

$$\|e^{t(K-I)}h\|_\infty = \|e^{t(L-I)}h\|_\infty \leq Ce^{-(1-\lambda)t}\|h\|_\infty.$$

This completes the proof. $\square$

Proof of Lemma 3.4. Using the standard mollification, for each $\gamma \in (0,1)$ we may choose functions $\bar{g}_\gamma \in C^2(\mathbf{R}^n)$ and $h_\gamma, H_\gamma \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$|\bar{g}_\gamma(x)| \vee |h_\gamma(x,\xi)| \leq C, \qquad |D\bar{g}_\gamma(x)| \vee \|D^2\bar{g}_\gamma(x)\| \vee |Dh_\gamma(x,\xi)| \leq C_\gamma,$$
$$|H_\gamma(p,\xi)| \vee |DH_\gamma(p,\xi)| \leq L_R,$$

for all $(x,p,\xi) \in \mathbf{R}^n \times B(0,R) \times I$ and $R > 0$ and for some constants $C > 0$, $C_\gamma > 0$,and $L_R > 0$. Here C does not depend on either γ or R, C_γ does not depend on R, but may depend on γ, etc. We may assume further that

$$(3.5) \qquad \begin{aligned} \int_I r(\xi)h_\gamma(x,\xi)d\xi = &\quad 0 \quad \text{for } x \in \mathbf{R}^n, \\ \int_I r(\xi)H_\gamma(p,\xi)d\xi = &\quad 0 \quad \text{for } p \in \mathbf{R}^n, \end{aligned}$$

$$g(x,\xi) \le \bar g_\gamma(x) + h_\gamma(x,\xi) \quad \text{and} \quad \bar g_\gamma(x) \le \bar g(x) + \sigma(\gamma) \quad \text{for all } (x,\xi) \in \mathbf{R}^n \times I,$$

where $\sigma(\gamma) \to 0$ as $\gamma \searrow 0$,

Fix $\gamma \in (0,1)$. In what follows we write $\bar g$ and h for $\bar g_\gamma$ and h_γ, respectively. This abuse of notation hopefully does not cause any confusion.

Fix $\varepsilon \in (0,1)$, and we define $f_\varepsilon \in C^1(\mathbf{R}^{n+1}) \otimes \mathcal{B}(I)$ by

$$f_\varepsilon(x,t,\cdot) = e^{\frac{t}{\varepsilon^2}(K-I)} h(x,\cdot) \quad \text{for } (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

Of course, we have

$$\begin{cases} \dfrac{\partial}{\partial t} f_\varepsilon(x,t,\xi) = & \frac{1}{\varepsilon^2}(K-I) f_\varepsilon(x,t,\cdot)(\xi) \\ f_\varepsilon(x,0,\xi) = & h(x,\xi) \end{cases}$$

for all $(x,t,\xi) \in \mathbf{R}^n \times \mathbf{R} \times I$. By Lemma 3.4, since (3.5) holds, we have

$$\begin{aligned} \|f_\varepsilon(x,t,\cdot)\|_\infty \le & \quad C_0 e^{-\frac{\delta t}{\varepsilon^2}} \|h(x,\cdot)\|_\infty \le C C_0 e^{-\frac{\delta t}{\varepsilon^2}}, \\ \|D f_\varepsilon(x,t,\cdot)\|_\infty \le & \quad \sqrt{n} C_\gamma C_0 e^{-\frac{\delta t}{\varepsilon^2}} \end{aligned}$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$, where δ and C_0 are positive constants from Lemma 3.5.

We set

$$\varphi_\varepsilon(x,\cdot) = S H_\varepsilon(D\bar g(x),\cdot)$$

in view of (3.5), and

$$w(x,t,\xi) = \bar g(x) + f_\varepsilon(x,t,\xi) + B_1 t + \varepsilon(\varphi_\varepsilon(x,\xi) + B_2) + \varepsilon B_3 (1 - e^{-\frac{\delta t}{\varepsilon^2}}) \tag{3.6}$$

for $(x,t,\xi) \in R_\infty \times I$, where B_1, B_2, and B_3 are positive constants to be fixed later. Recall that

$$(I-K)\varphi_\varepsilon(x,\cdot) = H_\varepsilon(D\bar g(x),\cdot) \quad \text{for } x \in \mathbf{R}^n,$$

and

$$D\varphi_\varepsilon(x,\cdot) = S\left(D^2 \bar g(x) D_p H_\varepsilon(D\bar g(x),\cdot)\right) \quad \text{for } x \in \mathbf{R}^n.$$

The last identity guarantees that

$$|D\varphi_\varepsilon(x,\xi)| \le C_1 \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I$$

for some constant $C_1 > 0$ independent of ε. We may assume as well that

$$|\varphi_\varepsilon(x,\xi)| \le C_1 \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

We calculate that

$$\begin{aligned}
J: &= w_t(x,t,\xi) - \frac{1}{\varepsilon}H(Dw(x,t,\xi),\xi) - \frac{\delta}{\varepsilon^2}\Big(\int_I k(\xi,\eta)w(x,t,\eta)d\eta - w(x,t,\xi)\Big) \\
&= \frac{1}{\varepsilon^2}(K-I)f_\varepsilon(x,t,\cdot)(\xi) + B_1 + \frac{\delta}{\varepsilon}B_3 e^{-\frac{\delta t}{\varepsilon^2}} \\
&\quad -\frac{1}{\varepsilon}H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) \\
&\quad -\frac{1}{\varepsilon^2}(K-I)\left(f_\varepsilon(x,t,\cdot) + \varepsilon\varphi_\varepsilon(x,\cdot)\right)(\xi) \\
&= B_1 + \frac{\delta}{\varepsilon}B_3 e^{-\frac{\delta t}{\varepsilon^2}} - \frac{1}{\varepsilon}H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) \\
&\quad +\frac{1}{\varepsilon}H(D\bar{g}(x),\xi).
\end{aligned}$$

Noting that as $\varepsilon \to 0$,

$$H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) = H(D\bar{g}(x),\xi) + O(\varepsilon + e^{-\frac{\delta t}{\varepsilon^2}}),$$

we see that

$$J \geq B_1 + \frac{\delta B_3}{\varepsilon}e^{-\frac{\delta t}{\varepsilon^2}} - M\Big(1 + \frac{\delta}{\varepsilon}e^{-\frac{\delta t}{\varepsilon^2}}\Big)$$

for some constant $M > 0$ which does not depend on ε.

We fix $B_1 = B_3 = M$, so that w is a viscosity supersolution of $(\mathrm{E})_\varepsilon$. Moreover, we fix $B_2 = C_1$ so that

$$u^\varepsilon(x,0,\xi) \leq w(x,0,\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

It is obvious that $w \in \mathcal{U}$. Thus, by a comparison theorem, we see that

$$u^\varepsilon(x,t,\xi) \leq w(x,t,\xi) \quad \text{for } (x,t,\xi) \in R_\infty \times I.$$

Sending $\varepsilon \searrow 0$, we see that

$$u^+(x,t) \leq \bar{g}_\gamma(x) + Mt \quad \text{for } (x,t) \in Q_\infty.$$

Writing $M(\gamma)$ for M in view of its dependence on γ and setting

$$\mu(t) = \inf\{\sigma(\gamma) + M(\gamma)t \mid \gamma \in (0,1)\} \quad \text{for } t \geq 0,$$

we get a modulus μ such that

$$u^+(x,t) \leq \bar{g}(x) + \mu(t) \quad \text{for } (x,t) \in Q_\infty.$$

Similar arguments ensure that for some modulus μ,

$$u^-(x,t) \geq \bar{g}(x) - \mu(t) \quad \text{for } (x,t) \in Q_\infty.$$

In case when $h = 0$, we use the same function w defined by (3.6) with $f_\varepsilon = 0$ and $B_2 = 0$ and argue in the same way as above, to conclude that

$$\bar{g}(x) - \mu(t) \leq u^-(x,t) \leq u^+(x,t) \leq \bar{g}(x) + \mu(t) \quad \text{for } (x,t) \in R_\infty$$

for some modulus μ. This completes the proof. $\square$

Lemma 3.6. *The functions u^+ and u^- are a viscosity subsolution and a viscosity supersolution of* (E)$_0$ *in Q_∞, respectively.*

We need the following lemma in the proof of Lemma 3.6.

Lemma 3.7. *There are a collection $\{H_\varepsilon\}_{\varepsilon\in(0,1)} \subset C^2(\mathbf{R}^n)\otimes\mathcal{B}(I)$ and a $(\{\omega_R\}_{R>0}, \{C_R\}_{R>0}) \in G_2$ such that for each $\varepsilon \in (0,1)$, H_ε satisfies* (H1) *and such that for all $(x,\xi) \in B(0,R)\times I$, $\varepsilon\in(0,1)$, and $R>0$,*

$$|H_\varepsilon(p,\xi) - H(p,\xi)| \le \omega_R(\varepsilon)\varepsilon, \quad |D_pH_\varepsilon(p,\xi) - D_pH_\varepsilon(p,\xi)| \le \omega_R(\varepsilon),$$
$$|H_\varepsilon(p,\xi)| \vee |D_pH_\varepsilon(p,\xi)| \le C_R, \quad \|D_p^2H_\varepsilon(p,\xi)\| \le \frac{\omega_R(\varepsilon)}{\varepsilon}.$$

Proof. By the standard mollification techniques, for each $\varepsilon > 0$ we find a function $H_\varepsilon \in \mathcal{D}_2 \cap C^2(\mathbf{R}^n)\otimes\mathcal{B}(I)$ such that for all $(x,\xi)\in B(0,R)\times I$, $\varepsilon\in(0,1)$, and $R>0$,

$$|H_\varepsilon(p,\xi) - H(p,\xi)| \le C_R\varepsilon, \quad |D_pH_\varepsilon(p,\xi) - D_pH(p,\xi)| \le \omega_R(\varepsilon),$$
$$|H_\varepsilon(p,\xi)| \vee |D_pH_\varepsilon(p,\xi)| \le C_R, \quad \|D_p^2H_\varepsilon(p,\xi)\| \le \frac{\omega_R(\varepsilon)}{\varepsilon},$$

where ω_R is a modulus and $C_R > 0$ is a constant, which can be chosen independently of ε.

Fix $R>0$ and fix such ω_R and C_R. Set

$$\sigma_R(r) = \inf\{(C_Rs)\vee\omega_R(sr)\vee\frac{\omega_R(sr)}{s} \mid 0<s<1\} \quad \text{for } r\ge 0.$$

Then it is clear that σ_R is a non-decreasing, upper semicontinuous, real-valued function on $[0,\infty)$ and that $\sigma_R(0)=0$.

By definition, for each $\varepsilon>0$ there is an $s\equiv s(\varepsilon)\in(0,1)$ such that

$$\sigma_R(\varepsilon)+\varepsilon > (C_Rs)\vee\omega_R(s\varepsilon)\vee\frac{\omega_R(s\varepsilon)}{s}.$$

Then the function $\widetilde{H}_\varepsilon(p,\xi) := H_{s\varepsilon}(p,\xi)$ and $\tilde{\sigma}_R(r) = \sigma_R(r)+r$ satisfy

$$|\widetilde{H}_\varepsilon(p,\xi) - H(p,\xi)| \le C_Rs\varepsilon \le \varepsilon\tilde{\sigma}_R(\varepsilon), \quad |D_p\widetilde{H}_\varepsilon(p,\xi) - D_pH(p,\xi)| \le \omega_R(s\varepsilon) \le \tilde{\sigma}_R(\varepsilon),$$
$$|\widetilde{H}_\varepsilon(p,\xi)| \vee |D_p\widetilde{H}_\varepsilon(p,\xi)| \le C_R, \quad \|D_p^2\widetilde{H}_\varepsilon(p,\xi)\| \le \frac{\omega_R(s\varepsilon)}{s\varepsilon} \le \frac{\tilde{\sigma}_R(\varepsilon)}{\varepsilon}$$

for all $(x,\xi)\in B(0,R)\times I$ and $\varepsilon>0$. In the above inequalities one may replace $\tilde{\sigma}_R$ by a modulus. Thus the collection $\{\widetilde{H}_\varepsilon\}_{\varepsilon\in(0,1)}$ together with appropriate choice of collections of moduli and of positive constants has the required properties. $\square$

Proof of Lemma 3.6. We begin by showing that u^+ is a viscosity subsolution of (E)$_0$. Let $\varphi \in C^3(Q_\infty)$, and assume that $u^+-\varphi$ attains a strict maximum at some point $(\hat{x},\hat{t})\in Q_\infty$.

Let $\{H_\varepsilon\}_{\varepsilon\in(0,1)}$ be a collection of functions from Lemma 3.7. For $\varepsilon \in (0,1)$, we define the function $\Phi(\cdot,\varepsilon)$ on $Q_\infty \times I$ by

$$\Phi(x,t,\xi,\varepsilon) = u^\varepsilon(x,t,\xi) - \varphi(x,t) - \varepsilon\varphi_1^\varepsilon(x,t,\xi) - \varepsilon^2\varphi_2^\varepsilon(x,t,\xi),$$

where

$$\begin{aligned}
\varphi_1^\varepsilon(x,t,\cdot) &= SH_\varepsilon(D\varphi(x,t),\cdot),\\
b^\varepsilon(x,t,\cdot) &= \langle D_pH_\varepsilon(D\varphi(x,t),\cdot), D\varphi_1^\varepsilon(x,t,\cdot)\rangle,\\
\bar{b}^\varepsilon(x,t) &= \int_I r(\xi)b^\varepsilon(x,t,\xi)d\xi,\\
\varphi_2^\varepsilon(x,t,\cdot) &= S\left(b^\varepsilon(x,t,\cdot) - \bar{b}^\varepsilon(x,t)\right)
\end{aligned}$$

for $(x,t) \in Q_\infty$.

Note that

$$\varphi_1^\varepsilon,\ b^\varepsilon,\ \varphi_2^\varepsilon \in C^1(Q_\infty)\otimes \mathcal{B}(I),$$

and

$$\begin{aligned}
D\varphi_1^\varepsilon(x,t,\cdot) &= S\left[D^2\varphi(x,t)D_pH_\varepsilon(D\varphi(x,t),\cdot)\right],\\
\frac{\partial}{\partial t}\varphi_1^\varepsilon(x,t,\cdot) &= S\left[\langle D\varphi_t(x,t), D_pH_\varepsilon(D\varphi(x,t),\cdot)\rangle\right]
\end{aligned}$$

for $(x,t) \in Q_\infty$.

Fix a compact neighborhood $V \subset Q_\infty$ of $(\hat{x},\hat{t})$. Using Lemma 3.7, we deduce that

$$\sup_{0<\varepsilon<1}\sup_{V\times I}\Big(|\varphi_1^\varepsilon| + |D\varphi_1^\varepsilon| + \left|\frac{\partial\varphi_1^\varepsilon}{\partial t}\right| + |b^\varepsilon| + |\varphi_2^\varepsilon|\Big) < \infty,$$

$$\sup_{V\times I}\Big(|D\varphi_2| + \left|\frac{\partial\varphi_2^\varepsilon}{\partial t}\right|\Big) \le \frac{\omega_V(\varepsilon)}{\varepsilon},$$

where ω_V is a modulus.

By the definition of u^+, there is a sequence $\varepsilon_j \searrow 0$ such that

$$\theta_j := \sup\{\Phi(x,t,\xi,\varepsilon_j) \mid (x,t)\in V,\ \xi \in I\} \to (u^+ - \varphi)(\hat{x},\hat{t}) \quad \text{as } j\to\infty.$$

Then we choose a sequence of points $(x_j,t_j,\xi_j) \in V \times I$ such that for each $j \in \mathbf{N}$, the function $\Phi(x,t,\xi_j,\varepsilon_j)$ attains a maximum over V at $(x_j,t_j) \in V$ and

$$\Phi(x_j,t_j,\xi_j,\varepsilon_j) \ge \theta_j - \varepsilon_j^3. \tag{3.7}$$

It is easily seen that

$$(x_j,t_j) \to (\hat{x},\hat{t}) \quad \text{as } j\to\infty.$$

Since u^ε is a viscosity subsolution of $(\mathrm{E})_\varepsilon$ in $Q_\infty \times I$, we have

$$\begin{aligned}
\varphi_t(x_j,t_j) \le\ & \frac{1}{\varepsilon_j}H(D\varphi(x_j,t_j) + \varepsilon_j D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j)\\
&+\varepsilon_j^2 D\varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j),\xi_j)\\
&+\frac{1}{\varepsilon_j^2}\Big(\int_I k(\xi_j,\eta)u^{\varepsilon_j}(x_j,t_j,\eta)d\eta - u^{\varepsilon_j}(x_j,t_j,\xi_j)\Big)\\
&+O(\varepsilon_j) \quad \text{as } j\to\infty.
\end{aligned} \tag{3.8}$$

Note that as $j \to \infty$,

$$
\begin{aligned}
& H(D\varphi(x_j,t_j) + \varepsilon_j D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j) + \varepsilon_j^2 D\varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j),\xi_j) \\
(3.9) \qquad & = H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) + \varepsilon_j \langle D_p H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j), D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j)\rangle + o(\varepsilon_j) \\
& = H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) + \varepsilon_j b^{\varepsilon_j}(x_j,t_j,\xi_j) + o(\varepsilon_j).
\end{aligned}
$$

From (3.7), we have

$$
\begin{aligned}
& u^{\varepsilon_j}(x_j,t_j,\xi) - u^{\varepsilon_j}(x_j,t_j,\xi_j) \\
& \le \varepsilon_j^3 + \varepsilon_j\left[\varphi_1^{\varepsilon_j}(x_j,t_j,\xi) - \varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j)\right] + \varepsilon_j^2\left[\varphi_2^{\varepsilon_j}(x_j,t_j,\xi) - \varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j)\right]
\end{aligned}
$$

for all $\xi \in I,\ j \in \mathbf{N}$. Hence, in view of the definition of the operator S, we have

$$
\begin{aligned}
& \int_I k(\xi_j,\eta)u^{\varepsilon_j}(x_j,t_j,\eta)d\eta - u^{\varepsilon_j}(x_j,t_j,\xi_j) \\
\le\ & \varepsilon_j^3 + \varepsilon_j(K-I)\varphi_1^{\varepsilon_j}(x_j,t_j,\cdot)(\xi_j) + \varepsilon_j^2(K-I)\varphi_2^{\varepsilon_j}(x_j,t_j,\cdot)(\xi_j) \\
=\ & \varepsilon_j^3 - \varepsilon_j H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) - \varepsilon_j^2\left[b^{\varepsilon_j}(x_j,t_j,\xi_j) - \bar{b}^{\varepsilon_j}(x_j,t_j)\right]
\end{aligned}
$$

Combining this with (3.8) and (3.9), we get

$$
(3.10) \qquad \varphi_t(x_j,t_j) \le \bar{b}^{\varepsilon_j}(x_j,t_j) + o(1) \quad \text{as } j \to \infty.
$$

Since

$$
\begin{aligned}
\bar{b}^{\varepsilon_j}(x_j,t_j) &= \int_I r(\xi)\langle D_p H(D\varphi(x_j,t_j),\xi), D^2\varphi(x_j,t_j)D_p a(D\varphi(x_j,t_j),\xi)\rangle d\xi + o(1) \\
&= \operatorname{tr}\left[\bar{A}(D\varphi(x_j,t_j))D^2\varphi(x_j,t_j)\right] + o(1)
\end{aligned}
$$

as $j \to \infty$, we conclude from (3.10) that

$$
\varphi_t(\hat{x},\hat{t}) \le \operatorname{tr}[\bar{A}(D\varphi(\hat{x},\hat{t}))D^2\varphi(\hat{x},\hat{t})],
$$

which shows that u^+ is a viscosity subsolution of $(\mathrm{E})_0$.

Arguments similar to the above prove that u^- is a viscosity supersolution of $(\mathrm{E})_0$. □

Proof of Theorem 3.3. In view of Lemma 3.4, we see that

$$
\lim_{r\searrow 0}\sup\{u^+(x,t) - u^-(y,s) \mid (x,t),(y,s) \in Q_r,\ |x-y| + |t-s| < r\} = 0.
$$

By Lemma 3.6, we know that u^+ and u^- are a viscosity subsolution and a viscosity supersolution of $(\mathrm{E})_0$. Thus, by using a comparison theorem, we see that $u^+ \le u \le u^-$ in Q_∞, from which we deduce easily that as $\varepsilon \searrow 0$,

$$
u^\varepsilon(x,t,\xi) \to u(x,t) \quad \text{locally uniformly in } Q_\infty \times I.
$$

Since $(\mathrm{E})_\varepsilon$ and $(\mathrm{E})_0$ are translation invariant in x, we conclude from the above that for any collection $\{y_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathbf{R}^n$, as $\varepsilon \searrow 0$,

$$
u^\varepsilon(x+y_\varepsilon,t,\xi) - u(x+y_\varepsilon,t) \to 0 \quad \text{locally uniformly in } Q_\infty \times I.
$$

Now a simple argument by contradiction shows that, for any $\delta \in (0,1)$, as $\varepsilon \searrow 0$,

$$u^\varepsilon(x,t,\xi) \to u(x,t) \quad \text{uniformly in } \mathbf{R}^n \times [\delta,\, \delta^{-1}] \times I. \tag{3.11}$$

Finally, if $g(x,\xi)$ is independent of ξ, then (3.11) and the last assertion of Lemma 3.4 yield the uniform convergence of $u^\varepsilon(x,t,\xi)$ to $u(x,t)$ in $R_T \times I$ for any $T \in (0,\infty)$ as $\varepsilon \searrow 0$. □

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